THE SCOTTISH CAFÉ PRINCIPLE

The best measure of how well you understand the material is how well you can explain it to a layman: TEACHING IS THE BEST FORM OF LEARNING.

DIVIDE A SQUARE. In how many different ways can you cut a square into four identical figures. Only straight cuts are possible and the entire square has to be divided. (ANSWER: infinitely many.)

THREE CARDS GAME. There are two players, Peter and Paul. Peter places three cards on a table in such a way that he knows which card is which. One of the cards is an ace and Paul's objective is to guess which card is an ace. Peter asks Paul to point to one of the cards. After Paul has pointed to a card, which remains covered, Peter uncovers one of the remaining two cards, one which is not an ace (note that this is always possible given that Peter knows precisely which card is where), and shows it to Paul. Two cards are now left on the table, one of them is an ace. Peter asks Paul if he wants to switch from his initial choice or not. This is Paul's final move. Paul has to decide which of the two cards he wants to be uncovered. If his final choice is an ace, Paul wins, otherwise he loses. Which action increases Paul's probability of winning? Switch, no-switch, or neither (i.e., it does not matter whether he switches or not) ? (ANSWER: Switching doubles the probability of winning—from 1/3 to 2/3.)

ULTIMATUM GAME. Two players, i and j. 100 million is to be divided between the two players. Only divisions in whole millions are possible. Player i proposes a division of 100 million to player j. Player j accepts or rejects the proposed division. If j accepts i's proposal, the proposed allocation becomes final. If i's proposal is rejected by j, the amount left to be divided is reduced to 90 million and the game goes into the second stage. In the second stage player j proposes the division of 90 million to player i. If i accepts the division, the allocation proposed becomes final. If i rejects j's proposal a division agreement is rendered impossible and both players get nothing. (ANSWER: (11,89) or (10,90))

WHY DO WE NEED A SCIENTIFIC THEORY:

(1) a theory (not necessarily a scientific one) is a way to look at reality
"divide a square" example

(2) we are hard wired to make certain cognitive mistakes
"three cards game" exemplifies a counterintuitive prediction

(3) our perception can be "blinded" by cultural/social values
"ultimatum game" exemplifies a non-obvious inference

(4) we need a "logical calculator" to carry out inferences even some very simple ones
"power in the system of international relations" example

PROPERTIES OF THE THREE EXAMPLES

(1) Objects used in the formulation of the problem have sharp connotations (a square, straight cut, identical figures, 100 million, etc.)

(2) Non-obvious or counterintuitive inferences. Each problem has a unique proper answer under assumptions/axioms of the theory in which it is formulated.

(3) "Divide a square" and "three cards game" are formulated in standard theories: Euclidean geometry and probability theory. "Ultimatum game" calls for a specification of axioms. Depending on what axioms are specified, different predictions are true.

FORMALIZATION OF THE ULTIMATUM GAME

NOTATION:

Players: 1, 2
By A denote a division $\{A_1, A_2\}$ in which player 1 gets A_1 and 2 gets A_2
Payoffs: $A_1, A_2 \in \{0, 1, 2, \dots, 100\}$.
Preference and indifference of player i=1,2 on divisions: $\succ_i$, $\sim_i$

AXIOMS (assumptions)

THE FIRST SET OF AXIOMS

Axiom 1. A is acceptable for i=1,2 if $A_i > 0$.

Axiom 2. For all divisions A, A^* and i=1,2, if $A_i > A_i^*$ then $A \succ_i A^*$.

Axiom 3. For all divisions A, A^*, i=1,2, and $j=\{1,2\}\backslash\{i\}$ if $A_i = A_i^*$ and $A_j > A_j^*$ then $A \succ_i A^*$.

Axiom 4. Players know each others' preferences.

Prediction implied by axioms 1-4: $A_1=11$, $B_2=89$.

THE SECOND SET OF AXIOMS

Axioms 1, 2, and 4 are the same. Axiom 3 is replaced by the following:

Axiom 3^*. For all divisions A, A^*, i=1,2, and $j=\{1,2\}\backslash\{i\}$ if $A_i = A_i^*$ and $A_j < A_j^*$ then $A \succ_i A^*$.

Prediction implied by axioms $1,2,3^*$ and 4: $A_i=10$, $B_j=90$.

NOTE: Other sets of axioms would generate other predictions.

THE FORM OF A SCIENTIFIC THEORY

EXAMPLES FROM EUCLIDEAN GEOMETRY

undefined (primitive) terms : point and line

axioms:
(examples)

Axiom 1. Every line is a collection of points
Axiom 2. There exist at least two points
Axiom 3. If *p* and *q* are points, then there exists one and only one line containing *p* and *q*.
Axiom 4. If *L* is a line, then there exists a point not on *L*.
Axiom 5. If *L* is a line, and *p* is a point not on *L*, then there exists one and only one line containing p that is parallel to *L*.

definitions:
(examples)

Definition 1. If a point *p* is an element of the collection of points which constitutes a line, then we say that *p is on L*.

Definition 2. Two lines are called *parallel* if there is no point which is on both of them.

theorems:
(an example)

Theorem 1. Every point is on at least two lines.

Proof: Let *p* denote any point. Since by Axiom 2 there exists at least two points, there must exist a point *q* distinct from *p*. And by Axiom 3 there exists a line *L* containing *p* and *q*. Furthermore, by Axiom 4 there exists a point *r* not on *L*, and (again by Axiom 3) a line *K* containing *p* and *r*. Now, by Axiom 1 every line is a collection of points. Hence for two lines to be distinct, the two collections must be different, i.e., one of them must contain a point that is not on the other. But this implies that lines *L* and *K* are distinct, since *K* contains the point *r* which is not on *L*. Moreover, since *p* is on both *L* and *K*, the theorem is proved.

THE INTERNATIONAL RELATIONS SYSTEM EXAMPLE

undefined terms: country and an alliance relation between two countries

notation:

By $C = \{c_1, \ldots, c_n\}$ denote a finite set of countries.
By A denote a binary (i.e., holding between two objects) relation between countries, i.e., $c_t A c_k$ means that c_t is allied with c_k.

axioms:

Axiom 1. For any two countries c_t and c_k, if c_t is an ally of c_k, then c_k is an ally of c_t. (If $c_t A c_k$ then $c_k A c_t$.)

definitions:

Definition 1. We will call a pair $<C, A>$, a *system of international relations.*
Definition 2. A *power* of a country in a system (of international relation) is a number of countries this country is allied with.

theorems:

Theorem 1. In any system with two or more countries there are at least two countries with the same power.

Proof: Suppose, by contradiction, that there is a system $<C, A>$ with n different countries ($n \geq 2$) where all countries have different powers. The only values that a power of a country can take in $<C, A>$ is one of the following n numbers: 0, 1, ... , n-1. Thus if all countries have different powers, then one of them must have power 0, another one, power 1, ..., and yet another one, power n-1. But if a country has power n-1, it must be allied with all other countries in the system, and this means that the system cannot contain a country with power 0. This, however, contradicts the fact that all countries in $<C, A>$ have different powers, and thus proves the theorem.

Theory of Preferences in Decision Making under Certainty

An inflated form of the preference theory

Undefined terms: alternatives, preference relation and indifference relation (both are binary relations on alternatives)

Notation: $D=\{x,y,z,...\}$ a finite set of alternatives; $\succ$ preference relation, $\sim$ indifference relation,
non stands for "it is not true that"

Axioms:

A1. (COMPARABILITY) For every x and y in D, $x \succ y$ or $y \succ x$ or $x \sim y$.
A2. (ASYMMETRY OF $\succ$) For every x, y in D, if $x \succ y$ then *non* $y \succ x$.
A3. (TRANSITIVITY OF $\succ$) For every x, y, and z in D, if $x \succ y$ and $y \succ z$ then $x \succ z$.
A4. (SYMMETRY OF $\sim$) For every x, y in D, if $x \sim y$ then $y \sim x$.
A5. (TRANSITIVITY OF $\sim$) For every x, y, and z in D, if $x \sim y$ and $y \sim z$ then $x \sim z$.

Preference theory proper (the shortest form i.e., the smallest number of undefined terms and axioms)

Axioms:

A1. (ASYMMETRY OF $\succ$) For every x, y in D, if $x \succ y$ then *non* $y \succ x$.
A2. (NEGATIVE TRANSITIVITY OF $\succ$) For every x, y, and z in D, if *non* $x \succ y$ and *non* $y \succ z$ then *non* $x \succ z$

DEFINITION
A binary relation on D is called a *(rational) preference relation* if it is asymmetric and negatively transitive.

THEOREM

If D is a finite set, a binary relation $\succ$ on D is a preference relation if and only if there exists a function u, from D into real numbers, (i.e., $u: D \rightarrow Re$) such that $x \succ y$ if and only if $u(x)>u(y)$.

Comment: The theorem means that it is possible to measure objects in the domain in the ordinal sense (on the ordinal scale) if and only if there is a preference relation on these objects. Thus asymmetry and negative transitivity are necessary and sufficient conditions for the existence of ordinal measurement on the objects in D.

THE FIRST, MOST GENERAL, NOTION OF RATIONALITY:

A decision maker is rational if his preference relation is asymmetric and negatively transitive.

Theory of Preferences in Decision Making under Uncertainty—Von Neumann-Morgenstern's Expected Utility Theory

Undefined terms: alternatives, preference relation on alternatives.
Theories subsumed: probability theory.

Notation: D={x,y,z,...} a finite set of alternatives; $\succ$ preference relation. We define the indifference relation ~ as follows: $x \sim y$ if and only if *non* $x \succ y$ and *non* $y \succ x$.

Axioms:

A1. $\succ$ is a preference relation.

A2. (INDEPENDENCE) For every x, y, and z in D, and any $p \in (0,1]$, if $x \succ y$ then $\left\langle \begin{matrix} p & x \\ 1-p & z \end{matrix} \right. \succ \left\langle \begin{matrix} p & y \\ 1-p & z \end{matrix} \right.$

A3. (CONTINUITY) For every x, y, and z in D, if $x \succ y \succ z$ then there is $p,q \in (0,1)$ such that

$$\left\langle \begin{matrix} p & x \\ 1-p & z \end{matrix} \right. \succ \quad y \quad \succ \left\langle \begin{matrix} q & x \\ 1-q & z \end{matrix} \right.$$

THEOREM

If D is a finite set, a binary relation $\succ$ on D satisfies axioms of the expected utility theory if and only if there exists a function u, from D into real numbers, (i.e., $u: D \rightarrow Re$) such that for every x,y in D and any $p \in [0,1]$

$$x \succ y \text{ if and only if } u(x) > u(y) \text{ and } u\left(\left\langle \begin{matrix} p & x \\ 1-p & y \end{matrix} \right. \right) = p\,u(x) + (1-p)\,u(y)$$

Comment: The theorem means that it is possible to measure objects in the domain in the cardinal sense (on the interval scale) if and only if $\succ$ satisfies axioms1-3. Thus axioms of the expected utility theory are necessary and sufficient conditions for the existence of cardinal (interval) measurement on the objects in D.

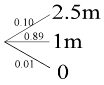
THE SECOND NOTION OF RATIONALITY:

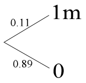
A decision maker is rational if his preference relation satisfies the axioms of the expected utility theory.

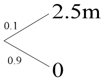
Two Applications of the Theorem

(1) Allais Paradox

In a series of studies Maurice Allais discovered that a considerable percentage of subjects have the following preferences over the four alternatives presented to them in the study:

1m $\succ$ 

and 

$\prec$ 

But from the Theorem above, if a decision maker who is ratiuonal has these two preferences, then there is a utility function such that:

(1) $u(1m) > 0.1\ u(2.5m) + 0.89\ u(1m) + 0.01\ u(0)$ (but this can be written as)
$u(1m) - 0.89\ u(1m) > 0.1\ u(2.5m) + 0.01\ u(0)$ (or, shorter as)
$0.11\ u(1m) > 0.1\ u(2.5m) + 0.01\ u(0)$

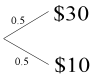
and

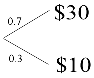
(2) $0.11\ u(1m) + 0.89\ u(0) < 0.1\ u(2.5m) + 0.9\ u(0)$ (but thiscan be written as)
$0.11u(1m) < 0.1\ u(2.5m) + 0.9\ u(0) - 0.89\ u(0)$ (or, shorter as)
$0.11\ u(1m) < 0.1\ u(2.5m) + 0.01\ u(0)$

But the last inequality in (1) is inconsistent with the last inequality in (2), hence we conclude that a rational decision maker (in the sense of the expected utility theory) cannot hold the two preferences.

(2) The Roomate Example

Suppose a friend of yours has the following two preferences. Does it mean that he/she is irrartional?

\$20 $\prec$ 

and \$20 $\succ$ 

From the Theorem above, if a decision maker who is ratiuonal has these two preferences, then there is a utility function such that:
(1) $u(\$20) < 0.5\ u(\$30) + 0.5\ u(\$10)$ and
(2) $u(\$20) > 0.7\ u(\$30) + 0.3\ u(\$10)$ and

But these twoinequalities are NOT inconsistent (take, for example, u(\$30)=0, u(\$20)=0.4, u(\$10)=1)! Hence, we conclude that a rational decision maker (as defined by the expected utility theory) can hold the two preferences.