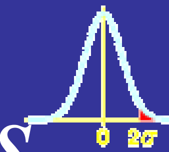
Probability—Counting Rules

SOCY601—Alan Neustadtl

Counting Rule 1: *For Sequences*



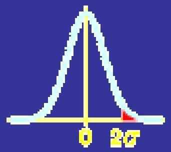
- *If $K_1, \ldots, K_n$ are the number of distinct events that can occur on trials 1, . . ., n in a series, then the number of different sequences of n events that can occur is $(K_1)(K_2) \ldots (K_n)$.*
- Example: The sociology department offers two sections of statistics with ten labs. How many possible sequences of the course are offered? $K_1 = 2$ for the number of lecture sections, and $K_2 = 10$ for the various labs. Therefore, there are 20 ways to sign up for the course—(2)(10) = 20
- Example: A travel agency offers special weekend trips to twelve cities with an option to travel by air, rail, or bus and an option for either luxury or budget accommodations. How many different ways can trips be arranged? $K_1 = 12$, $K_2 = 3$, and $K_3 = 2$ so, there are (12)(3)(2) = 72 different trip packages.

Counting Rule 2: *Number of Possible Sequences for n Trials*

- *If any of K mutually exclusive and exhaustive events can occur on each of n trials, then there are K^n different sequences that may result from a set of such trials.*
- Example: A test consists of fifteen multiple-choice questions, with each question having four possible answers. How many ways are there for a student to check off one answer for each question? Since $K_1 = K_2 = K_3 = \ldots K_{15} = 4$, there are altogether 4^{15} or (4)(4)(4)(4)(4) (4)(4)(4)(4)(4)(4)(4)(4)(4)(4) = 1,073,741,824 different ways for the answers to be checked off.
- Remember, only <u>one</u> of the 1,073,741,824 ways of answering has all the questions answered correctly and in 3^{15} or (3)(3)(3)(3)(3)(3)(3)(3)(3)(3)(3)(3)(3)(3)(3) = 14,348,907 of them, all the answers are wrong.

Counting Rule 3: *Permutations*

- *The Number of different ways that N distinct things may be arranged in order is $N! = (N)(N-1)(N-2) \ldots (3)(2)(1)$ (where $0!=1$).*
  - *An arrangement in order is called a permutation, so that the total number of permutations of N objects is N!. The symbol N! is called N factorial.*
- Example: The number of permutations of the four letters a, b, c, and d is 4!, or (4)(3)(2)(1) = 24. However, by rule one, the number of possible four-letter code words (if the letters may be repeated) is $4^4=256$.



Counting rule 4: *Ordered Combinations*

- *The number of ways of selecting and arranging r objects from among N distinct objects is* $\frac{N!}{(N-r)!}$
  This is often given as nPr.

- Example: There are 52 people eligible to run for president, vice president, secretary, and treasurer of a labor union. How many different ways can the members select their officers? Four objects (officers) are to be selected from among fifty-two distinct objects (all of the eligible members).

Counting rule 4: *Ordered Combinations*

- Example: There are 52 people eligible to run for president, vice president, secretary, and treasurer of a labor union. How many different ways can the members select their officers? Four objects (officers) are to be selected from among fifty-two distinct objects (all of the eligible members).

- This equals: $\binom{52}{4} = \frac{52!}{4!(52-4)!} = \frac{(52)(51)(50)(49)(\cancel{48})(\cancel{47})\ldots(\cancel{1})}{(\cancel{48})(\cancel{47})(\cancel{46})\ldots(\cancel{1})}$

- By Cancellation: $(52)(51)(50)(49) = 6{,}497{,}400$

Counting rule 4: *Ordered Combinations*

- Example: Determine the different permutations of two of the five vowels, *a*, *e*, *i*, *o*, and *u*. Using the formula, we have five distinct objects and we are taking them two at a time. So, there are $\frac{5!}{(5-2)!}$ or 20 ways to order the vowels.
- Another way of understanding this considers the number of choices at each step. There are five ways to select the first letter. After selecting the first letter, there are four ways of choosing the second. After that step, we do not care about the other letters. Therefore, there are (5)(4)=20 permutations of the vowels taken two at a time.

Counting rule 5: *Combinations*

- *The total number of ways of selecting r distinct combinations of N objects, regardless of order, is* $_{n}C_{r} = \binom{n}{r} = \frac{N!}{r!(N-r)!}$.

Counting rule 5: *Combinations*

- Example: Think back to the union election example. Fifty-two people were running for four different offices. We concluded that there are 6,497,400 ways to elect officers.

  Consider a member who has four people that he or she wants elected but does not care which person is elected to each office. That is, order does not matter. How many ways are there for four people (out of fifty-two) to be elected to four positions where all we care about is that they all get elected to some position? Using the formula we get:

$$\binom{52}{4} = \frac{52!}{4!(52-4)!} = \frac{(52)(51)(50)(49)}{(4)(3)(2)(1)}$$

$$= \frac{6,497,400}{24} = 270,725$$

Counting rule 5: *Combinations*

- Example: The director of a research laboratory needs to fill a number of research positions; two in biology and three in physics. There are seven applicants for the biology positions and 9 for the physicist positions. How many ways are there for the director to select these people?

  The two biologists can be selected in $\binom{7}{2}$ ways, the three physicists in $\binom{9}{3}$. So, by the multiplication rule, there are 1,764 ways for the director to hire people.

$$\left(\frac{7!}{2!(7-2)!}\right)\left(\frac{9!}{3!(9-3)!}\right)=\left(\frac{(7)(6)}{2}\right)\left(\frac{(9)(8)(7)}{6}\right)=(21)(84)=1,764$$

Using the Counting Rules: Poker Examples

- Many probability examples use the game of poker and the variety of hands that can be dealt from a deck of 52 cards. To calculate the probability associated with any particular hand, we first need to calculate how many hands can be dealt.

If the order of the cards is important we could calculate this number by applying the formula for permutations. There are 52 cards in a deck and we are interested in selecting 5 of them, or:

$$ {}_{52}P_5 = \frac{52!}{(52-5)!} = \frac{52!}{47!} = (52)(51)(50)(49)(48) = 311{,}875{,}200 $$

Poker Examples

- This does not really make sense because order does not matter when playing cards. That is, we really do not care that the cards in our hand are *A*, *K*, *Q*, *J*, and *10* of a suit, or whether they are *10*, *J*, *Q*, *K*, A. In this instance we should use combinations because order is *irrelevant*. This is expressed as:

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{(52)(51)(50)(49)(48)}{(5)(4)(3)(2)(1)} = 2{,}598{,}960$$

So, the probability of any one hand is equal to:

$$\frac{1}{2{,}598{,}960} = 0.00000038 \text{ or } 3.8 \text{ in } 10 \text{ million.}$$

Poker Examples

What is the probability of receiving a hand that is all spades? First, we must determine how many ways that we can select 5 spades. That is, given 13 spades, how many ways can we select 5 of them?

$$\binom{13}{5} = \frac{13!}{5!8!} = \frac{(13)(12)(11)(10)(9)}{(5)(4)(3)(2)(1)} = 1,287$$

Thus, the probability of a hand of all spades is:

$$\frac{\binom{13}{5}}{\binom{52}{5}} = \frac{1,287}{2,598,960} = 0.0005 \text{ or } 1 \text{ in } 2,000$$

Poker Examples

In poker, a flush is a hand of five cards all in the same suit. We have already determined the probability of a hand of all spades (which is one kind of flush). So, it is easy to determine the probability of a flush in any suit. The number of different flushes is equal to:

$$\frac{4\binom{13}{5}}{\binom{52}{5}} = \frac{(4)(1,287)}{2,598,960} = 0.002 \text{ or about 1 in 500}$$

Poker Examples

What about a poker hand of one pair with three different remaining cards? Having a pair means two aces, two kings, two queens, etc., of any suit. As a first step, of the four suits two must match. This is given as:

$$\binom{4}{2}$$

Since there are thirteen cards that can be used to make a pair, the number of pairs is equal to:

$$13\binom{4}{2}$$

The remaining *n* cards must show three of the twelve remaining numbers, but they may be of any suit. This number is given as:

$$\binom{12}{3}(4)(4)(4) \text{ or } \binom{12}{3}(4)^3$$

Poker Examples

Putting all of this together, we get:

$$\frac{13\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}} \approx 0.42$$

Poker Examples

We can use the same logic to determine the probability of receiving a full house. A full house consists of three of a kind and one pair. There are thirteen numbers that the three of a kind may have and then twelve possible numbers possible for the pair. The three of a kind must represent three of the four suits, and the pair, two of the four suits. This is expressed as:

$$13\binom{4}{3}12\binom{4}{2}$$

Given the total number of ways that we can deal five cards the probability of a full house is equal to

$$p(\text{full house}) = \frac{13\binom{4}{3}12\binom{4}{2}}{\binom{52}{5}} \approx 0.0014$$

Poker Examples

Finally, we can determine the probability of a flush, or five cards all in the same suit. There are exactly four suit possibilities, and five cards out of thirteen numbers that would give us a flush. So, the different number of flushes is:

$$4\binom{13}{5}$$

Therefore, the probability of a flush with five cards is equal to:

$$\frac{4\binom{13}{5}}{\binom{52}{5}} \approx 0.00198$$